SAMPLE-DEPENDENT PHASE TRANSITIONS IN DISORDERED EXCLUSION MODELS

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Abstract

We give numerical evidence that the location of the first order phase transition between the low and the high density phases of the one dimensional asymmetric simple exclusion process with open boundaries becomes sample dependent when quenched disorder is introduced for the hopping rates.

Key words: phase transition, asymmetric simple exclusion process, disordered systems, open system, stationary non-equilibrium state

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The effect of quenched disorder on phase transitions is one of the most studied and best understood aspects of the theory of disordered systems [1, 2, 3, 4, 5, 6, 7, 8, 9]. For equilibrium systems, the Harris criterion [4, 5] allows one to decide whether the critical behavior of a second order phase transition is altered by a weak disorder. One also knows [6, 7, 8, 9] that first order transitions are suppressed by a weak disorder in low enough dimension.

It is now well established that non equilibrium systems can exhibit phase transitions even in one dimension (for a review see [10]). For simple systems as the asymmetric exclusion with open boundaries (the TASEP), the phase diagram is known exactly [11, 12, 13, 14, 15]. It is then a natural question to investigate what is the effect of disorder on such a phase diagram [16, 5].

The one dimensional asymmetric simple exclusion process (ASEP)[17, 18, 19] has been considered in several contexts ranging from biopolymerisation and electrophoresis to the study of car traffic [20]. It is also related to other problems of statistical physics like surface growth and directed polymers in random media [21, 22]. It describes a one dimensional lattice of L sites, each site of which is either empty or occupied by a single particle (exclusion rule). During each infinitesimal time interval dt, a particle at site i attempts to move to its right neighboring site with probability pdt and the move is successful only if the target site $i+1$ is empty. At the boundary (site 1 and site L) the dynamics is modified as follows (open boundary conditions): during each infinitesimal time interval dt, a particle is injected on site 1 of the lattice with probability αdt if this site is empty; and if a particle is present on site L, it is removed with probability βdt. When the boundary parameters α and β are varied, the model exhibits several phase transitions [11, 12, 14, 13, 15]: second order phase transitions along the lines $\alpha = p/2 < \beta$ (from a low density phase to a maximal current phase) and $\beta = p/2 < \alpha$ (from a high density phase to a maximal current phase), and a first order transition from the low to the high density phase along the line

$$\alpha = \beta < \frac{p}{2} \tag{1}$$

The goal of the present work is to study the effect of quenched disorder on this first order phase transition. Different ways of introducing quenched disorder have been considered for the ASEP. The most studied is the *particlewise* disorder [23, 16, 24]. It corresponds to the case where the hopping rates p depend on the particle which attempts to jump. One can think of it as a model of traffic along a one lane road where each vehicle has a different, random velocity. This model can be exactly solved through a mapping to the zero-range process [23].

We consider here the less studied *sitewise* disorder, where to each lattice site is associated a random, quenched hopping rate p_i. In the traffic

analogy, this would correspond to the case where localized accidents or bottlenecks locally decrease the mean speed of vehicles. Even the adjunction of a single localized defect in the hopping rate p_i transforms the ASEP into a difficult, and yet unsolved problem [25, 26, 27]. There are only few analytical results in the case of site disorder, the most noticeable ones being the existence of an hydrodynamic limit for the density, the concavity of the current of particles as a function of the density of particles [28] and the reflection invariance of the current [29]. Monte-Carlo simulations on ASEP on a ring geometry (i.e. with periodic boundary conditions)[30, 31] have shown that the current-density relation is modified by quenched sitewise disorder, with the appearance of a plateau between two densities ρ_c and $1-\rho_c$ where the current becomes independent of the density of particle ρ on the ring. Tripathy and Barma [30] have also shown that this plateau corresponds to a *segregated-density regime* where the disorder induces a phase separation between a high density and a low density phase. Outside this plateau (i.e. for densities $\rho < \rho_c$ or $\rho > 1-\rho_c$), the steady state profile is uniform on a macroscopic scale (but with microscopic variations). Some bounds for ρ_c have been obtained [16]. When the distribution of the p_i allows values as close as possible to 0 a third regime occurs, in which the steady state current vanishes in the thermodynamic limit [16, 30] .

In the open boundaries case, when particles are injected at site 1 and removed at site L on the last one, one expects the system to have the same 3 phases (low density, high density and maximal current) as in the pure case. The question then is how the nature and the location of the phase transitions are modified by the effect of disorder.

Here we investigate numerically the effect of disorder on the first order transition (1). We choose a binary distribution of the bulk hopping rates p_i.

$$p_i = \begin{cases} p_{\min} \text{ with probability } \frac{1}{2} \\ p_{\max} \text{ with probability } \frac{1}{2} \end{cases} \tag{2}$$

with $p_{\min} = 0.8$, $p_{\max} = 1.2$.

In equilibrium systems (with short range inteactions), the properties of finite systems depend smoothly on external parameters and phase transitions usually occur in the thermodynamic limit, i.e. in the limit of an infinite system size. The ASEP on a finite chain with or without disorder is a Markov process with a stationnary state which also depends smoothly on the parameters α and β and one needs to take the limit of an infinite size to observe phase transitions.

As the p_i's depend on i, one needs to choose a procedure to add new sites to a given sample in order to take the thermodynamic limit. In the following we compare two different procedures:

- Procedure C (C for center): we increase the size of a given sample by adding new sites at the center of the sample. So as L increases, the p_i close to the boundaries remain unchanged and sites with new p_i's are introduced at the center, i.e. further and further from the boundaries

- Procedure B (B for boundary): we add the new sites at the two boundaries, whereas the center of the sample remains unchanged.

For each sample of size L, we make Monte Carlo simulations for different values of α at fixed $\beta = 0.15$ and different values of α. Figure 1 shows the steady state density $\langle \frac{N}{L} \rangle$ where N is the total number of particles in the system as a function of α. The density is averaged over typicaly 3×10^5 updates per site. Figure 1(a) corresponds to three different samples when the size is increased according to procedure (C); figure 1(b) to a single sample when the size is increased by the sides (procedure (B)), and figure 1(c) shows the pure case ($p_i = 1$). As α varies, one sees a transition from a low density for small α to a high density for large α. The slope at the crossing point becomes steeper and steeper when the size increases; the position of the value of α where the slope is maximum is sample dependent but size independent in figure 1(a) when the size is increased by the center. On the contrary it is size dependent in figure 1(b) when the size is increased by the sides.

In the same way, we show in figures (2) the variance $(\langle N^2 \rangle - \langle N \rangle^2)/L$ as the size is increased. In the pure case (figure 2(c)), the location of the maximum of the curve converges to $\alpha_c = \beta$, the phase transition point of the infinite system as the size increases. When the size of the disordered samples is increased by the center (figure 2(a)), the location of the maximum of the curve converges in a way similar to the pure case but to a critical value for α which depends on the initial sample. This suggests that in the thermodynamic limit, the system undergoes a phase transition for a critical value α_c which is sample dependent and depends on the hopping rates near boundaries. Moreover, as the size dependence of the maximum of the curve is the same in figure 1(a) and 1(c), one expects that the transition remains first order in presence of disorder.
When the size of the sample is increased by the sides (figure 2(b)), the location of the maximum does not converge in the thermodynamic limit.

Figure 3 shows the distribution of the pseudo transition point (the value of α for which the variance in figure 2 is maximum) for quenched systems of size $L = 50$ and $L = 100$. We generated 40000 samples for each size and each sample was simulated over 300000 steps per site, after a sufficiently long transient time to reach the steady state. The histogram is then obtained by discretizing the α_c-axe in boxes of length 4×10^{-4}. In order

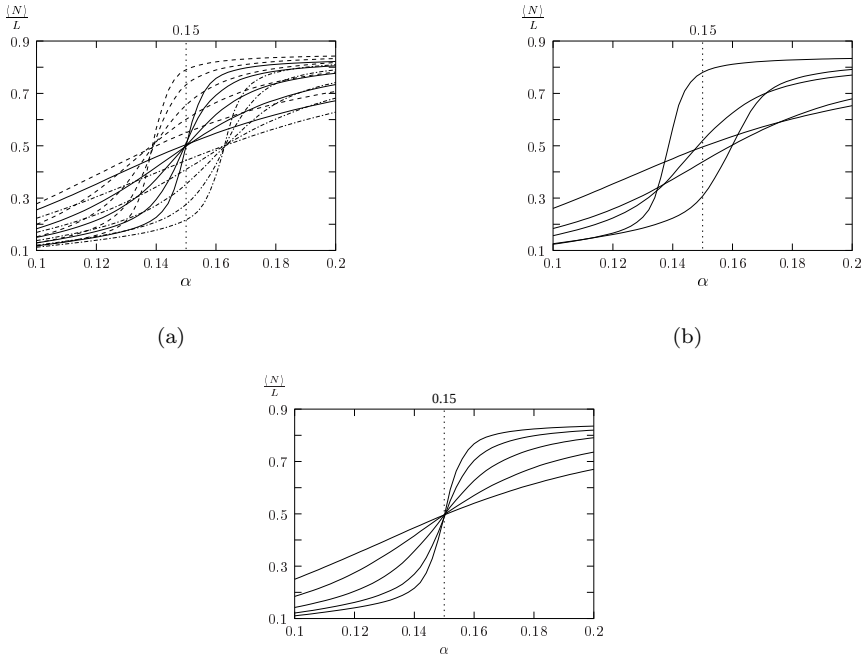

Figure 1: Mean density $\frac{\langle N \rangle}{L}$ versus α for a sample of size 11, 21, 41, 81 and 161. In figure 1(a), the system size is increased by adding new sites at the center (procedure C), and 3 samples are shown. In figure 1(b), the size is increased by adding new sites on the sides (procedure B) and a single sample is shown. In figure 1(c), the pure case (all the $p_i = 1$) is shown for comparison. The dotted line shows on the three figures the transition point of the pure TASEP.

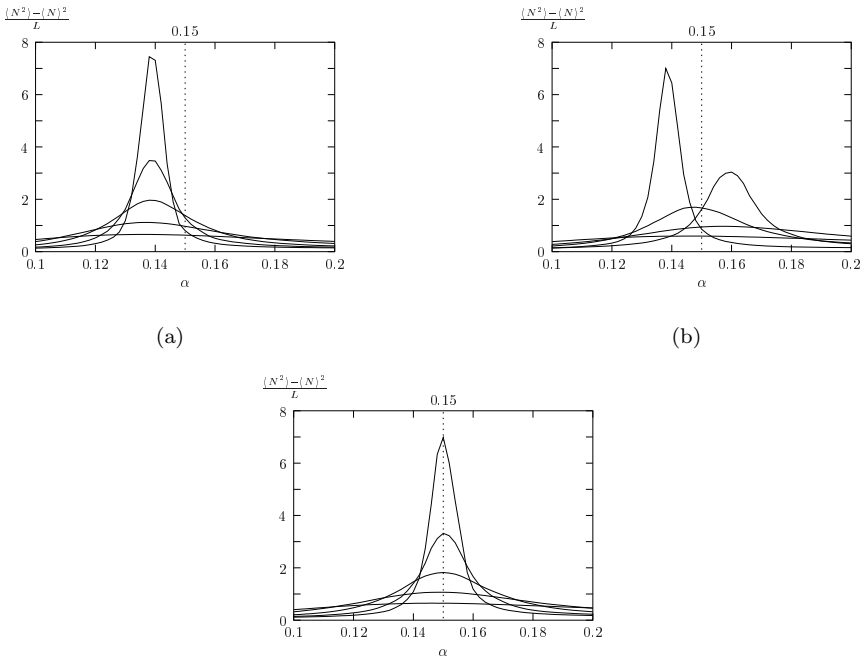

Figure 2: Variance $(\langle N^2\rangle - \langle N\rangle^2)/L$ versus α for a single sample of size 11, 21, 41, 81 and 161. In figure 2(a), the system size is increased by the center (procedure C). In figure 2(b), the size is increased by the sides (procedure B). In figure 2(c), the pure case (all the $p_i = 1$) is shown for comparison. The dotted line shows the position of the transition point for the pure TASEP.

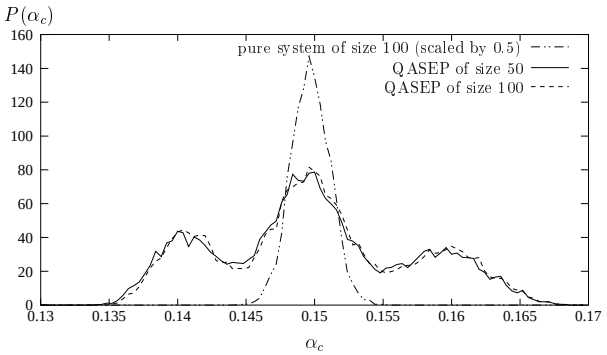

Figure 3: Distribution $P(\alpha_c)$ of the pseudo transition point α_c for $\beta = 0.15$ when the hopping rates p_i are given by (2) (full line: size $L = 50$; dashed line: size $L = 100$). The distribution does not shrink as the size increases. The distribution (divided by 2) of the pseudo transition point for the pure system measured via the same method is shown for comparison.

to estimate the numerical noise on the determination of the critical-point coming from the finite time length (3×10^5) of our simulation, we include the distribution obtained over 20000 samples of the pure system (the distribution for the pure case has been divided by 2 to make the top of the peak visible in figure 3).

The distribution in presence of disorder remains unchanged as the system size L increases by a factor 2, which suggests that the same distribution would be observed in the thermodynamic limit. As the distribution $P(\alpha_c)$ in presence of disorder is much broader than in the pure case, its width is not due to the limited time of the measurement. Still, all the details of the distribution $P(\alpha_c)$ cannot be resolved on a scale smaller than the width of the $P(\alpha_c)$ of the pure system. Therefore longer simulations should reveal more structure at smaller scales of the distribution $P(\alpha_c)$ (while the distribution for the pure case should become narrower).

Figure 4 tries to evaluate the correlation between the value of α_c and the hopping rates close to the boundaries. We note by $\lfloor . \rfloor_{\alpha_c}$ the average over the samples that have a critical point α_c (up to the discretization of the α_c-axis). On figures (4) we show the average of some hopping rates $\lfloor p_i \rfloor_{\alpha_c}$, namely the first p_1, the second p_2 and the last p_{L-1}, as a function of the value of the critical point α_c. One sees that the three maxima of the distribution of the critical points can be characterized by the value of the first hopping rate p_1 and p_{L-1}: the left peak corresponds to $p_1 = p_{\max}$, $p_{L-1} = p_{\min}$, the right peak to $p_1 = p_{\min}$, $p_{L-1} = p_{\max}$. The curve of the average of the second hopping rate $\lfloor p_2 \rfloor_{\alpha_c}$ as a function of α_c supports the idea that the value of p_2 has a weaker influence on α_c than p_1 or p_{L-1}.

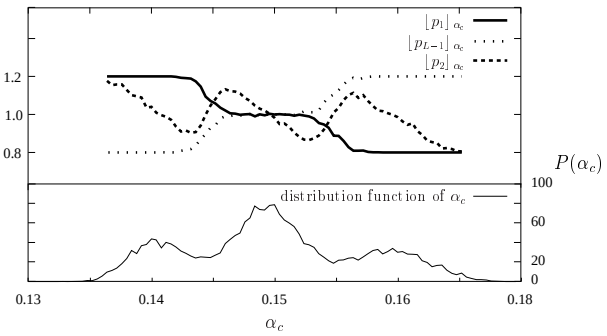

Figure 4: $\lfloor p_1 \rfloor_{\alpha_c}$, $\lfloor p_2 \rfloor_{\alpha_c}$ and $\lfloor p_{L-1} \rfloor_{\alpha_c}$ versus the position of the critical point α_c. The full curve is the same distribution of the critical point α_c as in figure 3. We see that all samples which have a pseudo transition point α_c in the left peak have $p_1 = p_{max}$ and $p_{L-1} = p_{min}$ and all the sample which have a pseudo transition point α_c in the right peak have $p_1 = p_{min}$ and $p_{L-1} = p_{max}$.

Therefore the picture which emerges is that α_c is sample dependent and the influence of the p_i's on α_c decreases with their distance to the boundary (which could lead to a fractal $P(\alpha_c)$ (smoothed out in our figure 3 due to the finite time length of our simulation)). The fact that α_c is strongly correlated to the p_i close to the boundaries explains why α_c converges when procedure C is used as in figure 1(a) and why it wanders around when procedure B is used.

From a technical point of view, let us mention that, for each sample, we made the measurement for all the values of α in a single simulation. To do so, we introduced K different classes of particles [32, 33], indexed from 1 to K, and whose dynamics is the same as in the TASEP, except that the particles of a higher class K behave like holes in front of particles of lower class K' (when $K' < K$). More precisely, we still impose the exclusion rule, so there is no more than one particle on each site of the system; and at each infinitesimal time dt, a particle on site i attempts to jump to its right neighbor with probability $p_i dt$. The jump succeeds only if the target site $i+1$ is empty or contains a particle of a class higher than the particle which attempts the jump. If the jump succeeds, the particles on site i and $i+1$ are exchanged.
To simulate K values $\alpha_1 < \alpha_2 < \cdots < \alpha_K$ of α the rule is to introduce particles of class k at rate $\alpha_k - \alpha_{k-1}$ (with $\alpha_0 = 0$) at the left boundary provided that the first site is empty or occupied by a particle of higher class. At the right boundary, a particle present at site L is removed at rate β irrespective of its class.

To do our measurement of the number of particles for a given choice of α_k, we count as particles all particles of class $\leq k$ and as holes all particles of class $\geq k+1$.

In this work, we have given numerical evidence that for the ASEP with open boundaries, *the location of the first order phase transition* between the low and high density phases *becomes sample-dependent* in presence of quenched sitewise disorder. This seems to be a property specific to non-equilibrium systems and probably to the one dimensional case as the free energy of equilibrium systems (with short range interactions) is self averaging and the distribution of the pseudo critical point shrinks as the system size increases [34].

As in the pure case the transition is controled by the boundary parameters α and β, our observation that the location of the phase transition is influenced by the hopping rates on the sites close to the boundaries is not a surprise. In higher dimension, there is a large number of sites closest to the boundaries and so their effect should be averaged out.

It would be interesting to see how the other properties of the phase diagram of the TASEP would be affected by site disorder: how the second order transitions are modified by the disorder, how the domain wall (uniformly distributed over the sample in the pure case [35]) is distributed in the disordered case when one sits along at the first order transition point.

It would be also interesting to see whether these sample dependent phase transitions could be seen in other non-equilibrium one dimensional systems [36, 37] when disorder is introduced.

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